

Lens space surgeries on A'Campo's divide knots

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Dedicated to Professor Takao Matumoto on the occasion of his 60th birthday.

Abstract

It is proved that every knot in the major subfamilies of J. Berge's lens space surgery (i.e., knots yielding a lens space by Dehn surgery) is presented by an L-shaped (real) plane curve as a *divide knot* defined by N. A'Campo in the context of singularity theory of complex curves. For each knot given by Berge's parameters, the corresponding plane curve is constructed. The surgery coefficients are also considered. Such presentations support us to study each knot itself, and the relationship among the knots in the set of lens space surgeries.

1 Introduction

If r/s Dehn surgery on a knot K in S^3 yields the lens space $L(p, q)$, we call the pair $(K, r/s)$ a *lens space surgery*, and we also say that K admits a lens space surgery, and that r/s is the *coefficient* of the lens space surgery. The task of classifying lens space surgeries, especially knots that admit lens space surgeries has been a focal point in low-dimensional topology and has been invigorated of late by results from the Heegaard Floer homology theories of Ozsváth–Szábo [OSz] (see also [He], [Ta] and so on). Before the first hyperbolic examples found by Fintushel–Stern [FS] in 1980, only torus knots (Moser [Mo]) and their 2-cables (Bailey–Rolfsen [BR]) were known. After [FS], some more examples were found (see [Ma]). In 1990, Berge [Bg] pointed out a “mechanism” of known lens space surgery, that is, *doubly-primitive knots* in the Heegaard surface of genus 2. Berge also gave a conjecturally complete list of such knots, described them by Osborne–Stevens's “R-R diagrams” in [OST], and classified such knots into three families, and into 12 types in detail:

- (1) *Knots in a solid torus* (Type I, II, ... and VI)

Dehn surgery along a knot in a solid torus whose resulting manifold is also a solid torus. This family was studied in [Bg2].

- (2) *Knots in genus-one fiber surface* (Type VII and VIII)

Dehn surgery along a knot in the genus-one fiber surface (of the right/left-handed trefoil (Type VII) or of figure eight (Type VIII)), see [Ba, Ba3] and [Y1].

- (3) *Sporadic examples (a), (b), (c) and (d)* (Type IX, X, XI and XII, respectively)

⁰2000 *Mathematics Subject Classification*: Primary 57M25, 14H50 Secondary 55A25.

Keywords: Dehn surgery, plane curves

¹This work was partially supported by Grant-in-Aid for Scientific Research No.18740029, Japan Society for the Promotion of Science.

Their surgery coefficients are also decided. Thus we call them *Berge's knots* of lens space surgery, or *Berge's lens space surgeries*. The numbering VII, \dots and XII are also used in the recent works by Baker in [Ba2, Ba3]. It is conjectured by Gordon [Go1, Go2] that every knot of lens space surgery is a doubly-primitive knot. Berge has claimed that his list of doubly-primitive knots is complete (i.e., any doubly-primitive knot belongs to (1), (2) or (3)), but it has not appeared.

In the present paper, we are concerned with the family (1). Its subfamily Type I consists of torus knots. Type II consists of 2-cables of torus knots. Their presentations as A'Campo's divide knots are already studied in [GHY] and [Y2]. Thus our targets are Type III, IV, V and VI.

Notation. Throughout the paper, we let the symbol \mathcal{X} denote one of these Types, i.e., $\mathcal{X} = \text{III, IV, V or VI}$.

To describe the knots in each Type \mathcal{X} , in [Bg2], Berge defined five parameters $\delta, \varepsilon \in \{\pm 1\}$ and $A, B, b \in \mathbf{Z}$ (They satisfy some certain conditions depending on \mathcal{X}). We introduce two new parameters k, t such that B, b are uniquely calculated from k, t and vice-versa. By $K_{\mathcal{X}}(\delta, \varepsilon, A, k, t)$, we mean the knot defined by the parameters in Type \mathcal{X} . (Type VI is slightly different from the others.) Taking opposite δ corresponds to the mirror image of the knot. Note that, if a lens space surgery (K, r) belongs to Type \mathcal{X} , $(K!, -r)$ is also a lens space surgery and belongs to the same Type \mathcal{X} , where $K!$ is the mirror image of K . See Section 2 for details on the parameters.

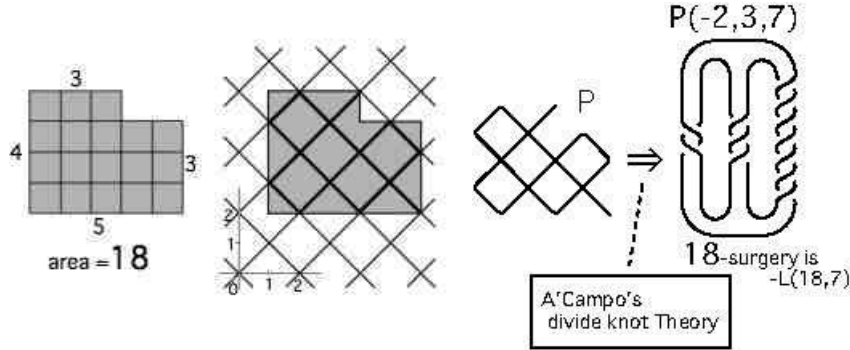


Figure 1: Pretzel knot $(-2, 3, 7)$ with coefficient 18

The theory of A'Campo's *divide knots and links* comes from singularity theory of complex curves. The *divide* is (originally) a relative, generic immersion of a 1-manifold in a unit disk in \mathbf{R}^2 . A'Campo [A1, A2, A3, A4] formulated the way to associate to each divide C a link $L(C)$ in S^3 . In the present paper, we regard a PL (piecewise linear) plane curve as a divide by smoothing the corners. The class of divide links properly contains the class of the links arising from isolated singularities of complex curves, i.e., positive torus knots, and iterated torus knots satisfying certain inequalities in their parameters.

Definition 1.1 Let X be the $\pi/4$ -lattice defined by $\{(x, y) | \cos \pi x = \cos \pi y\}$ in xy -plane (\mathbf{R}^2). By an *L-shaped region*, we mean a union of two rectangles sharing a corner and overlapping along an edge of one, where rectangles are assumed that all edges are parallel to either

x -axis or y -axis, and that all vertices are at lattice points ($\in \mathbf{Z}^2$). We call a plane curve an *L-shaped curve* if the curve P is obtained as an intersection $X \cap \mathcal{L}$ of X and an L-shaped region \mathcal{L} . We define $area(P)$ of an L-shaped curve $P = X \cap \mathcal{L}$ as the area (2-dim. volume) of the L-shaped region \mathcal{L} defining P .

See Figure 1. It is the starting example of our results. The L-shaped curve $P = X \cap \mathcal{L}$, as a divide, presents the pretzel knot of type $(-2, 3, 7)$. Its 18 surgery is a lens space, which is one of the examples in [FS]. Note that the area of P is equal to 18, the coefficient of the lens space surgery. Our main result is:

Theorem 1.2 *Up to mirror image, every Berge's knot of lens space surgery in Type III, IV, V and VI is one of A'Campo's divide knots, and can be presented by an L-shaped curve.*

In fact, for the given parameters \mathcal{X} and $(\delta, \varepsilon, A, k, t)$, we will construct an L-shaped curve $P_{\mathcal{X}}(\varepsilon, A, k, t) = X \cap \mathcal{L}_{\mathcal{X}}(\varepsilon, A, k, t)$, see Demonstration in Subsection 4.4. Note that opposite δ corresponds to the mirror image.

Theorem 1.3 *Our L-shaped curve $P_{\mathcal{X}}(\varepsilon, A, k, t)$ presents the Berge's knot $K_{\mathcal{X}}(\delta, \varepsilon, A, k, t)$ in Type \mathcal{X} ($\mathcal{X} = \text{III, IV, V or VI}$), up to mirror image:*

$$L(P_{\mathcal{X}}(\varepsilon, A, k, t)) = K_{\mathcal{X}}(1, \varepsilon, A, k, t), \text{ or its mirror image } K_{\mathcal{X}}(-1, \varepsilon, A, k, t).$$

One of $K_{\mathcal{X}}(\pm 1, \varepsilon, A, k, t)$ is presented by a positive braid (say w) and the other is by a negative one (the inverse w^{-1}). The divide knot $L(P_{\mathcal{X}}(\varepsilon, A, k, t))$ is exactly equal to the positive one, but the choice at δ (1 or -1) depends on \mathcal{X} and (ε, A, k, t) .

Next, we study the surgery coefficients. By the Cyclic Surgery Theorem of Culler–Gordon–Luecke–Shalen [CGLS], if a hyperbolic knot K admits a lens space surgery, then the coefficient is integral. By $\text{coef}(K_{\mathcal{X}}(\delta, \varepsilon, A, k, t))$, we denote the surgery coefficient of the lens space surgery of the knot as in Type \mathcal{X} . Note that there exist some hyperbolic knots that have two coefficients of lens space surgery (such coefficients are proved to be consecutive in [CGLS]), and belong to different Types as the pairs with the coefficients. It is the reason why we state “as in Type \mathcal{X} ”.

Theorem 1.4 *Under the correspondence in Theorem 1.3, the area of the L-shaped curve P is equal to (the absolute value of) the coefficient of the lens space surgery of $L(P)$ as in Type \mathcal{X} , or is greater by one:*

$$area(P) - |\text{coef}(L(P))| = 0 \text{ or } 1.$$

This theorem will be proved as Lemma 5.3, in which we will decide the choice (0 or 1) by the parameters. We will prove that $\text{coef}(L(P)) > 0$ in Lemma 5.1. Thus we will change $|\text{coef}(L(P))|$ to $\text{coef}(L(P))$ in Lemma 5.3.

Theorem 1.2 can be proved by combination of Lemma 2.1 and Lemma 3.8. But the aim of the present paper is the construction of the L-shaped curves using the operation *adding squares* on L-shaped curves in Section 4, and studying the knots and the family of knots by them. Some knots are obtained from other knots (possibly in other Types) by some twistings. By our method adding squares, we can search such pairs, and check such relations easily, see Section 6.

Here we survey on divide presentation of the other Berge's knots (in Type VII and later), shortly. All knots are considered up to mirror image. Type VII consists of the knots that the author [Y1] gave L-shaped curve presentations first. Type VIII contains some knots that is hard (in the author's opinion) to decide whether it is a divide knot or not, and (if it is) to present by a divide. The author has shown that every sporadic knot (in Type IX and later) is a divide knot and has shown a method to construct the divide. But he does not know whether it can be presented by an L-shaped curve or not.

Note that there exist a family of L-shaped divide knot whose $\text{area}(P)$ -surgery is not a lens space [Y3], but such L-shaped divide knots tend to have exceptional Dehn surgeries, to the author's knowledge [Y2, Y4].

This paper is organized as follows: In the next section, we review Berge's knots and their parameters in detail. In Section 3, we review A'Campo's divide knot theory and define L-shaped curves. In Section 4, developing a method *adding squares*, we construct L-shaped curves (regions) for Berge's knots. In Section 5, we will prove Theorem 1.3 and Lemma 5.3, the precise version of Theorem 1.4. Finally, in Section 6, we study some applications, advantages to present Berge's knots as divide knots. We place Tables 1 and 2 after the reference list for the reader's convenience.

2 Berge's knots of Type III, IV, V and VI

We recall the Berge's parametrization of knots in Type \mathcal{X} . We use his original parameters $\delta, \varepsilon, A, B, b$ and a constant a ($:= 0$ or 1) defined in [Bg2], and introduce two new parameters k and t .

We start with the following:

- (1) δ and ε are signs ($\in \{\pm 1\}$). The opposite δ corresponds to the mirror image.
 - \cdot A is a positive integer, whose range and parity (even or odd) depends on \mathcal{X} ,
 - \cdot k runs in $\mathbf{N}_{\geq 0}$. B is decided by (ε, A, k) and satisfies that $0 < 2A \leq B$.
 - \cdot $b, t \in \mathbf{Z}$. They can be negative.
- (2) The parameters k, t are introduced instead of the conditions in [Bg2] written by sentences and by congruences, respectively. For example, instead of “ $(B + \varepsilon)/A$ is an odd integer” in [Bg2, Table 3(p.15)], we set $B = A(3 + 2k) - \varepsilon$. Instead of “ $b \equiv -2\varepsilon\delta A \pmod{B}$ ” in [Bg2, Table 3(p.15)], we set $b = -\delta\varepsilon(2A + tB)$. These are the relations between (B, b) and (k, t) in Type III. They are similar in other Types, but slightly different, see Table 1(1).
- (3) The independent parameters are $(\delta, \varepsilon, A, k, t)$ in Type III, IV and V, but is (δ, A, t) in Type VI. We formally regard the latter as $(\delta, \varepsilon, A, k, t) = (\delta, -1, A, 0, t)$, i.e., we fix $\varepsilon := -1, k := 0$ in Type VI, for the convenience.

Notation. By $K_{\mathcal{X}}(\delta, \varepsilon, A, k, t)$, we denote the knot parametrized as $(\delta, \varepsilon, A, k, t)$ in Type \mathcal{X} , by Berge in [Bg2].

Now we go into the detail. See Table 1(1), (2) and (3). In Table 1(1), we define B and b (depending on \mathcal{X}), using a temporary parameter l . For fixed A and ε , the possible values of l

are in an arithmetic sequence, depending on \mathcal{X} . We parametrize the sequence by $k \in \mathbf{N}_{\geq 0}$ as in Table 1(2). In every case, the surgery coefficient is $bB + \delta A$, where B depends on ε, A, k . In Table 1(3), we deform the coefficients into the form including the terms $+kA^2 + tB^2$ (or $+k(2A)^2 + tB^2$ in Type III). These are related to our method adding squares in Section 4. Note that, if a knot K with coefficient r belongs to Type \mathcal{X} , its mirror image $K^!$ with $-r$ (i.e., opposite δ) also belongs to the same Type \mathcal{X} .

Using these parameters, in [Bg2], Berge has already given the braid presentations of these knots:

Lemma 2.1 (Berge [Bg2]) *Every knot $K_{\mathcal{X}}(\delta, \varepsilon, A, k, t)$ is presented as the closure of the braid $W(B)^b W(A+1-a)^\delta$ of index B , where $W(n) = \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1$, see Figure 3.*

Definition 2.2 We define an anti-homomorphic (i.e., $\rho(\beta_1 \beta_2) = \rho(\beta_2) \rho(\beta_1)$) involution π -rotation ρ on the braid group of index n by extending $\rho(\sigma_i) = \sigma_{n-i}$, see Figure 2. By $\beta' \stackrel{\rho}{=} \beta$, we mean $\beta' = \rho(\beta)$ and equivalently $\rho(\beta') = \beta$.

In this opportunity, we define another notation $\beta' \sim \beta$ as that the closure of β' is the same knot or link to that of β . Note that $\beta' \stackrel{\rho}{=} \beta$ up to conjugate implies $\beta' \sim \beta$.

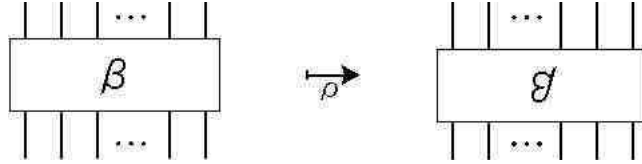


Figure 2: π -rotation ρ

Our L-shaped divide knots are always presented by positive braids, see Section 3, while any divide knot is a closure of a strongly quai-positive braid (Lemma 3.2(7)). Thus, first, if $b < 0$, we take the mirror image (i.e., change the sign δ , then b becomes to $-b > 0$) and next, we use the following lemma if it is necessary.

Lemma 2.3 *Let a_1, a_2 and c be positive integers with $a_1 < a_2$. The closure of the braid $W(a_2)^c W(a_1)^{-1}$ of index a_2 is the same knot to that of $W(a_2)^{c-1} W(a_2 - a_1 + 1)$.*

Proof.

$$\begin{aligned} W(a_2)W(a_1)^{-1} &= \sigma_{a_2-1} \sigma_{a_2-2} \cdots \sigma_{a_1} \cdots \sigma_2 \sigma_1 (\sigma_{a_1-1} \cdots \sigma_2 \sigma_1)^{-1} \\ &= \sigma_{a_2-1} \sigma_{a_2-2} \cdots \sigma_{a_1} \\ &\stackrel{\rho}{=} W(a_2 - a_1 + 1). \end{aligned}$$

Since π -rotation ρ is anti-homomorphic, and the braid $W(n)$ of index n ($= a_2$) is fixed by ρ , we have

$$\begin{aligned} W(a_2)^c W(a_1)^{-1} &= W(a_2)^{c-1} \cdot W(a_2) W(a_1)^{-1} \\ &\stackrel{\rho}{=} W(a_2 - a_1 + 1) W(a_2)^{c-1}, \end{aligned}$$

which is conjugate to $W(a_2)^{c-1} W(a_2 - a_1 + 1)$. \square

Figure 3 illustrates Lemma 2.3. $K_{\text{III}}(1, -1, 2, 0, 0)$ is presented by the braids $W(7)^4 W(3)$, and $K_{\text{III}}(-1, 1, 2, 0, 0)$ is presented by $W(7)^4 W(3)^{-1} \sim W(7)^3 W(5)$.

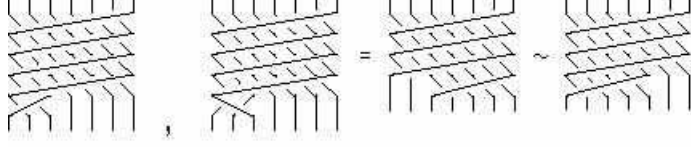


Figure 3: $W(7)^4W(3)$ and $W(7)^4W(3)^{-1} \sim W(7)^3W(5)$

3 L-shaped curves and A’Campo’s divide knots

The theory of A’Campo’s *divide knots and links* [A1, A2, A3, A4] comes from singularity theory of complex curves. It is a method to associate to each divide (a plane curve) C a link $L(C)$ in the 3-dimensional sphere S^3 . The original definition of divide knots in [A1] is differential-geometric. Hirasawa [Hi] visualized the construction. We are concerned with the plane curves of special type, called “L-shaped curves”, see Subsection 3.2. For such special curves, we can use another method introduced by Couture–Perron [CP], see Subsection 3.3.

3.1 A’Campo’s divide knots

We start with the typical example of divide knots, see Figure 4:

Lemma 3.1 (Goda–Hirasawa–Y [GHY], see also [AGV, Gu]) *Let a, b be a pair of positive integers and $\mathcal{R}(a, b)$ be an $a \times b$ -rectangle region. A plane curve defined by $X \cap \mathcal{R}(a, b)$ (a billiard curve of type $B(a, b)$) presents the torus link $T(a, b)$ as a divide.*

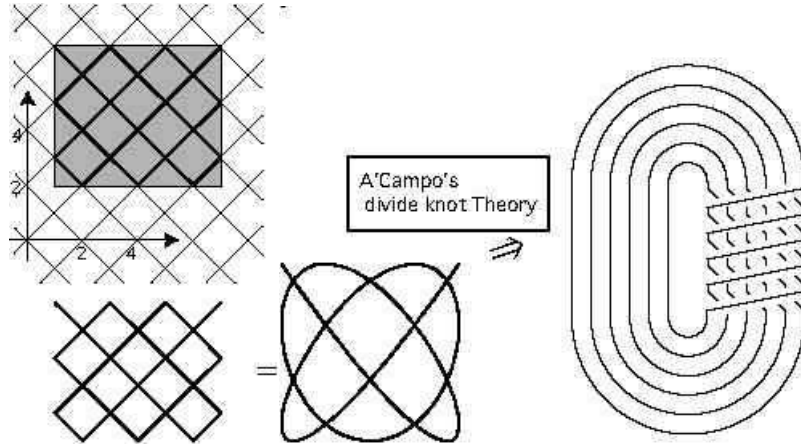


Figure 4: A billiard curve presents a torus knot (ex. $T(6, 5)$)

Some characterizations of (general) divide knots and links are known, and some topological invariants $L(P)$ can be gotten from the divide P directly. Here, we list some of them.

Lemma 3.2 ((1)–(6) by A’Campo [A2], (7) by Hirasawa [Hi], Rudolph [R])

- (1) $L(P)$ is a knot (i.e., connected) if and only if P is an immersed arc.

- (2) If $L(P)$ is a knot, the unknotting number, genus and 4-genus of $L(P)$ are all equal to the number $d(P)$ of the double points of P .
- (3) If $P = P_1 \cup P_2$ is the image of an immersion of two arcs, then the linking number of the two component link $L(P) = L(P_1) \cup L(P_2)$ is equal to the number of the intersection points between P_1 and P_2 .
- (4) If P is connected, then $L(P)$ is fibered.
- (5) A divide P and its mirror image $P!$ present the same knot or link: $L(P!) = L(P)$.
- (6) If P_1 and P_2 are related by some Δ -moves, then the links $L(P_1)$ and $L(P_2)$ are isotopic: If $P_1 \sim_{\Delta} P_2$ then $L(P_1) = L(P_2)$, see Figure 5.
- (7) Any divide knot is a closure of a strongly quasi-positive braid, i.e., a product of some σ_{ij} in Figure 5.



Figure 5: Basics on divide knots

For theory of divide knots, see also Rudolph's "C-link" in [R] and [C, HW].

3.2 Preliminary on L-shaped curves

First, we parametrize L-shaped regions by four positive integers a_1, a_2, b_1, b_2 that satisfy $a_1 < a_2$ and $b_1 < b_2$, see Figure 6:

Definition 3.3 (L-shaped region at the origin) In xy -plane, we define

$$L[a_1, a_2; b_1, b_2] := \{(x, y) | 0 \leq x \leq a_1, 0 \leq y \leq b_2\} \cup \{(x, y) | 0 \leq x \leq a_2, 0 \leq y \leq b_1\}.$$

By *concave corner*, we mean the point (of the region) at the coordinate (a_1, b_1) in the definition above. We will call not only $L := L[a_1, a_2; b_1, b_2]$ but also its transformations $\mathcal{L} = T(L)$ an *L-shaped region of type* $[a_1, a_2; b_1, b_2]$, where T is a transformation in xy -plane generated by the reflection r_X along the x -axis (Lemma 3.2(5)), the rotation R by $\pi/2$ about a lattice point and the parallel translation $+\vec{n}$ by a lattice point $\vec{n} (\in \mathbf{Z}^2)$.

Let X be the $\pi/4$ -lattice defined by $\{(x, y) | \cos \pi x = \cos \pi y\}$ in xy -plane. A lattice point $(m, n) (\in \mathbf{Z}^2)$ is called *even* (or *odd*, resp.) if $m + n$ is even (or odd). We are concerned only with the case that the intersection $X \cap \mathcal{L}$ is the image of a generic immersed arc. Thus, we always control $+\vec{n}$ and assume that

- (*) The concave point of an L-shaped region \mathcal{L} is placed at an odd point.

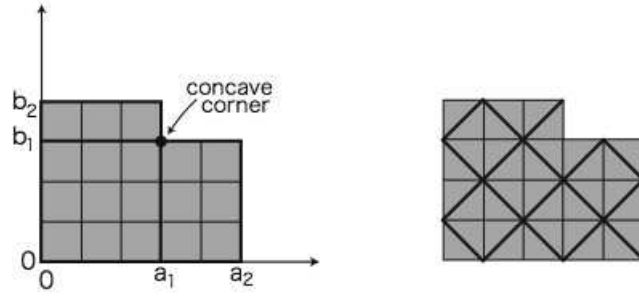


Figure 6: L-shaped region $L[3, 5; 3, 4]$

Assuming (*), the parameter $[a_1, a_2; b_1, b_2]$ defines a unique plane curve up to isotopy, i.e., it depends on neither r_X, R nor translations keeping even/odd points. We call the corresponding plane curve an *L-shaped curve of type* $[a_1, a_2; b_1, b_2]$. Of course, for an L-shaped curve P of type $[a_1, a_2; b_1, b_2]$, we have

$$\text{area}(P) = a_2 b_1 + a_1 b_2 - a_1 b_1.$$

On the other hand, the number $d(L)$ of double points of L is

$$d(P) = \{a_2(b_1 - 1) + b_2(a_1 - 1) - a_1 b_1 + 1\}/2,$$

because double points are the even points of the interior of the L-shaped region.

The condition (*) is not sufficient for $X \cap \mathcal{L}$ to be the image of an immersed arc. In fact, it possibly consists of multiple components or contains some circle components.

The following proposition follows from Lemma 3.2(2).

Proposition 3.4 *If the L-shaped curve $P = X \cap \mathcal{L}$ of type $[a_1, a_2; b_1, b_2]$ with the assumption (*) is an immersed arc, then the genus $g(L(P))$ of the divide knot $L(P)$ is (the unknotting number, and the 4-genus are also) equal to the number $d(P)$ of the double points of P :*

$$g(L(P)) = \{a_2(b_1 - 1) + b_2(a_1 - 1) - a_1 b_1 + 1\}/2.$$

Thus, it holds that $\text{area}(P) - 2g(L(P)) = a_2 + b_2 - 1$.

3.3 L-shaped divide knots

In [CP] Couture and Perron pointed out a method to get the braid presentation from the divide (the plane curve) in the restricted cases, called “ordered Morse” divides. Our L-shaped curves are all ordered Morse, thus we can apply their method. It is a special case of Hirasawa’s method in [Hi].

Lemma 3.5 *The divide link presented by the L-shaped curve of type $[a_1, a_2; b_1, b_2]$ is the closure of the braid $W(a_2)^{b_1} W(a_1)^{b_2 - b_1}$ of index a_2 , where $W(n) = \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1$.*

Such a link should be regarded as a “rationally twisted” torus link in the following sense: The link is obtained by a “ $(b_2 - b_1)/a_1$ twist” of the parallel a_1 strings in a_2 strings of torus link $T(a_2, b_1)$ in the standard position $W(a_2)^{b_1}$.

Example 3.6 The divide knot presented by the L-shaped curve $[3, 5; 3, 4]$ is the closure of the braid $(\sigma_2\sigma_4\sigma_1\sigma_3)^3\sigma_2\sigma_1$ (conjugate to $(\sigma_4\sigma_3\sigma_2\sigma_1)^3\sigma_2\sigma_1$) of index 5, which is $P(-2, 3, 7)$.

Proof. First, we define the words $o(n)$ and $e(n)$ in the braid group of index a_2 as follows:

$$e(n) := \prod_{i: \text{ even}, i < n} \sigma_i, \quad o(n) := \prod_{i: \text{ odd}, i < n} \sigma_i,$$

where n is a positive integer less than or equal to the index. Note that σ_i and σ_j are commutative if i and j have the same parity. If $j \leq k < l$ and k is even, $o(k)^{-1}o(l)$ is a product of σ_i 's with $i \geq k + 1$, thus is commutative with both $o(j)$ and $e(j)$. Similarly, if $j \leq k < l$ and k is odd, then $e(k)^{-1}e(l)$ is commutative with both $o(j)$ and $e(j)$.

In the case of L-shaped curves, Couture-Perron's method is summarized as the algorithm in Figure 7. By direct application of the algorithm to the L-shaped curve of type, we have:

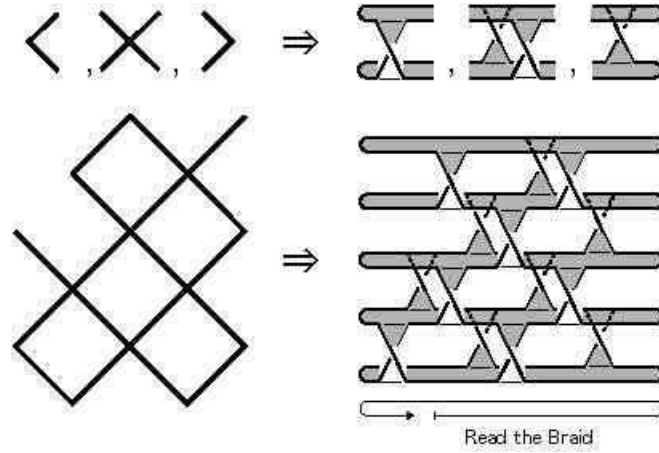


Figure 7: Couture-Perron's method in the case $[3, 5; 3, 4]$ ($\pi/2$ rotated)

Claim 1 [CP] The L-shaped curve of type $[a_1, a_2; b_1, b_2]$ presents the closure of the braid

$$B[a_1, a_2; b_1, b_2] := \begin{cases} \left(e(a_2)o(a_2) \right)^{b_1} \left(e(a_1)o(a_1) \right)^{b_2-b_1} & \text{if } a_1 \text{ is odd,} \\ \left(o(a_2)e(a_2) \right)^{b_1} \left(o(a_1)e(a_1) \right)^{b_2-b_1} & \text{if } a_1 \text{ is even.} \end{cases}$$

The key idea of the rest of the proof is in Figure 8. Let $G(n)$ be as follows.

$$G(n) := \begin{cases} e(n+1)o(n)e(n-1)o(n-2)e(n-3) \cdots e(4)o(3) & \text{if } n \text{ is odd,} \\ o(n+1)e(n)o(n-1)e(n-2)o(n-3) \cdots e(4)o(3) & \text{if } n \text{ is even.} \end{cases}$$

It is a product of some σ_i 's with $i < n$.

Claim 2

$$\begin{aligned} G(n-2)^{-1}e(n)o(n) G(n-2) &= W(n) & \text{if } n \text{ is odd,} \\ G(n-2)^{-1}o(n)e(n) G(n-2) &= W(n) & \text{if } n \text{ is even.} \end{aligned}$$

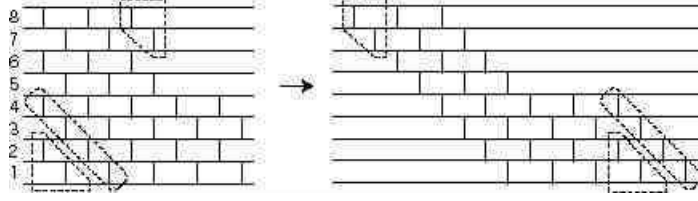


Figure 8: Braid of L-shaped curve $[5, 9; 3, 5]$

In fact, if n is odd, $e(n-1)^{-1}e(n) = \sigma_{n-1}$ and it commutes with $o(n-2)^{-1}$. Next, $o(n-2)^{-1}o(n) = \sigma_{n-2}$ and it commutes with $e(n-3)^{-1}$. We repeat such reductions until $o(3)^{-1}o(5) = \sigma_3$ inductively, and end with $e(4)o(3) = \sigma_2\sigma_1$. The other case is proved similarly.

Next, we set $H(a_1, a_2) := \rho(G(a_2 - a_1 - 1))$, where ρ is π -rotation in Definition 2.2. Then, $H(a_1, a_2)$ is a product of some σ_i 's with $i > a_1 + 1$. Thus we have

Claim 3 $H(a_1, a_2)$ commutes with $e(a_1)$, $o(a_1)$ and $G(a_1 - 2)$.

Let $\Omega(a_1, a_2) := H(a_1, a_2)^{-1}G(a_1 - 2)$. By Claim 2 and 3, It holds

Claim 4

$$\begin{aligned} \Omega(a_1, a_2)^{-1}e(a_1)o(a_1)\Omega(a_1, a_2) &= W(a_1) & \text{if } a_1 \text{ is odd,} \\ \Omega(a_1, a_2)^{-1}o(a_1)e(a_1)\Omega(a_1, a_2) &= W(a_1) & \text{if } a_1 \text{ is even.} \end{aligned}$$

The following is the most troublesome step.

Claim 5

$$\begin{aligned} \Omega(a_1, a_2)^{-1}e(a_2)o(a_2)\Omega(a_1, a_2) &= W(a_1)W(a_2)W(a_1)^{-1} & \text{if } a_1 \text{ is odd,} \\ \Omega(a_1, a_2)^{-1}o(a_2)e(a_2)\Omega(a_1, a_2) &= W(a_1)W(a_2)W(a_1)^{-1} & \text{if } a_1 \text{ is even.} \end{aligned}$$

To prove Claim 5, we divide the braid of index a_2 into two parts, lower and higher parts along the a_1 -th string. Here we denote $a_2 - a_1 + 1$ by $\overline{a_1}$. In the case a_1 is odd,

$$e(a_2)o(a_2) = \begin{cases} e(a_1)o(a_1)\rho(e(\overline{a_1})o(\overline{a_1})) & \text{if } a_2 \text{ is odd,} \\ e(a_1)o(a_1)\rho(o(\overline{a_1})e(\overline{a_1})) & \text{if } a_2 \text{ is even.} \end{cases}$$

In the former case, by Claim 3, the conjugation of $e(a_2)o(a_2)$ by $\Omega(a_1, a_2)$ is divided as the product of that of $e(a_1)o(a_1)$ by $G(a_1 - 2)$ and that of $\rho(e(\overline{a_1})o(\overline{a_1}))$ by $H(a_1, a_2)^{-1}$. Since $H(a_1, a_2)$ is defined as $\rho(G(\overline{a_1} - 2))$, it holds

$$\begin{aligned} H(a_1, a_2)\rho(e(\overline{a_1})o(\overline{a_1}))H(a_1, a_2)^{-1} &= \rho(G(\overline{a_1} - 2)^{-1}e(\overline{a_1})o(\overline{a_1})G(\overline{a_1} - 2)) \\ &= \rho(W(\overline{a_1})) \\ &= W(a_2)W(a_1)^{-1} \end{aligned}$$

Here, we use Claim 2 with odd $n = \overline{a_1}$. In the other cases, including a_2 is even, the proofs are similar.

Finally, by Claim 1, 4 and 5,

$$\begin{aligned}\Omega(a_1, a_2)^{-1} B[a_1, a_2; b_1, b_2] \Omega(a_1, a_2) &= (W(a_1)W(a_2)W(a_1)^{-1})^{b_1} W(a_1)^{b_2-b_1} \\ &= W(a_1)W(a_2)^{b_1} W(a_1)^{-1} W(a_1)^{b_2-b_1}\end{aligned}$$

It is conjugate to $W(a_2)^{b_1} W(a_1)^{b_2-b_1}$. The proof of Lemma 3.5 is complete. \square

By the symmetry between the L-shaped curve of type $[a_1, a_2; b_1, b_2]$ and that of type $[b_1, b_2; a_1, a_2]$, we have an extension of the well-known symmetry $T(b, a) = T(a, b)$ of torus knots.

Corollary 3.7 *The closures of the braids*

$$W(b_2)^{a_1} W(b_1)^{a_2-a_1} \text{ of index } b_2 \quad \text{and} \quad W(a_2)^{b_1} W(a_1)^{b_2-b_1} \text{ of index } a_2$$

define the same link.

Lemma 3.8 *Let a_1, a_2 and c be positive integers with $a_1 < a_2$, and δ be a sign ($\in \{\pm 1\}$). Then, the knot of the closure of the braid of type $W(a_2)^{\pm c} W(a_1)^\delta$ of index a_2 is presented as a divide knot presented by an L-shaped curve, up to mirror image:*

- (++) *The knot $W(a_2)^c W(a_1)$ is presented by the L-shaped curve $[a_1, a_2; c, c+1]$.*
- (+-) *The knot $W(a_2)^c W(a_1)^{-1}$ is presented by the L-shaped curve $[a_2 - a_1 + 1, a_2; c-1, c]$.*
- (--) *The knot $W(a_2)^{-c} W(a_1)^{-1}$ is the mirror image of the knot presented by the L-shaped curve $[a_1, a_2; c, c+1]$.*
- (-+) *The knot $W(a_2)^{-c} W(a_1)$ is the mirror image of the knot presented by the L-shaped curve $[a_2 - a_1 + 1, a_2; c-1, c]$.*

Proof. The case (++) in the lemma follows from Lemma 3.5 directly, and (+-) follows from Lemma 2.3 and Lemma 3.5. The cases (--) and (-+) follow from (++) and (+-) respectively, since, if a knot K is the closure of the braid w , then the mirror image $K!$ is that of the inverse w^{-1} , in general. \square

By Lemma 2.1 and Lemma 3.8, Theorem 1.2 is already proved: Up to mirror image, every Berge's knot of lens space surgery in Type III, IV, V and VI is one of A'Campo's divide knots, and can be presented by an L-shaped curve.

We end this section with referring the fiberedness of Berge's knots. By Theorem 1.2 and fiberedness of divide knots in Lemma 3.2(4), we can show

Corollary 3.9 (Teragaito [Te], Ozsváth–Szábo[OSz]) *Every Berge's knot of lens space surgery in Type III, IV, V and VI is fibered.*

This corollary can be proved by Lemma 2.1, 2.3 and the fact that knots presented by positive (or negative) braids are fibered [S]. In fact, Teragaito [Te] (see [HM, §5.7]), has shown that every Berge's knots (including Type VII, ..., XII, see Section 1) is fibered, by proving the braid positivity. Ozsváth–Szábo [OSz, §5] also proved fiberedness of every Berge's knot from another view points.

4 Construction of L-shaped curves

We define the operation *adding squares* on L-shaped curves (via L-shaped regions), its drawing notations, and explain how to construct the L-shaped curves $P_{\mathcal{X}}(\varepsilon, A, k, t) = X \cap \mathcal{L}_{\mathcal{X}}(\varepsilon, A, k, t)$ for Berge's knots in Type \mathcal{X} , given by the parameters $(\delta, \varepsilon, A, k, t)$. From now on, we consider only L-shaped divide knots, i.e., the case that the L-shaped curve P (with the assumption $(*)$ in Section 3.2) is the image of an immersed arc (Lemma 3.2(1)).

4.1 Adding squares I

We start with the following:

Definition 4.1 For a positive integer n , we call the operation on L-shaped curves, changing from that of type $[a_1, a_2; b_1, b_2]$ to $[a_1, a_2 + nb_1; b_1, b_2]$ or $[a_1 + nb_2, a_2 + nb_2; b_1, b_2]$ *adding n squares*, see examples in Figure 9. As a drawing notation, we specify the edge along which the squares are added, and write n near the edge. By the symmetry in Corollary 3.7, we also call the changing from $[a_1, a_2; b_1, b_2]$ to $[a_1, a_2; b_1, b_2 + na_2]$ or $[a_1, a_2; b_1 + na_2, b_2 + na_2]$ adding n squares.

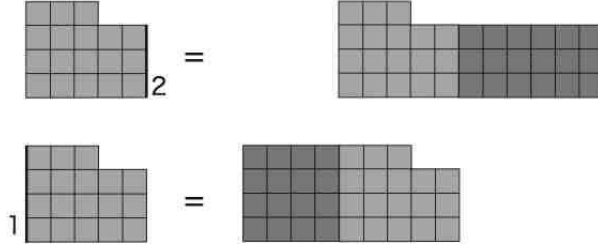


Figure 9: Adding squares I

Lemma 4.2 *Adding n squares on an L-shaped curve P along an edge (of the region) corresponds to positive (i.e., right-handed) n full-twists on the divide knot $L(P)$ along the unknot defined by the edge.*

This lemma is proved by using the braid presentation in Lemma 3.5. Note that a full-twist is in the center of the braid group.

Adding a square can be regarded as “blow-down”, in the following sense. The coordinate transformation $(x, y) = (x, xt)$ (or $= (yt, y)$) is called a *blow up* in singularity theory, and is used for resolution of singularity of complex curves, see [HKK, p.16]. For example, for a coprime positive integers (p, q) with $p < q$, the complex curve $x^q - y^p = 0$ becomes to $x^p(x^{q-p} - t^p) = 0$ by the transformation. In this example, the link of the singularity at the origin changes from the torus knot $T(p, q)$ to $T(p, q - p)$ and an unknot defined by the complex line t -axis ($x = 0$ in xt -plane) appears. The unknot is the axis of the full-twist. It corresponds to the Kirby calculus in the bottom figure in Figure 10, i.e., (-1) -framed unknot appears and the other components change by a left-handed full-twist along the unknot. Framings also change by a certain formula. See [K, GS] for Kirby calculus.

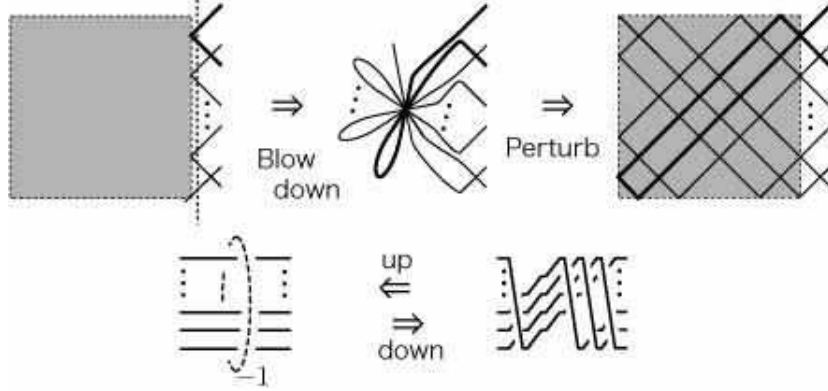


Figure 10: Adding a square corresponds to a full-twist

We call its inverse operation, i.e., the transformation $(x, y) = (x, y'/x)$ a blow down. (For example, $y^2 = x + \epsilon$ becomes $y'^2 = x^2(x + \epsilon)$, where both hand-sides are multiplied by x^2 .) Let C be a real plane curve in xy -plane that intersects with y -axis transversely. By the transformation, the all intersection points C and y -axis concentrate to the origin of xy' -plane, and the left half ($x < 0$) of C turns upside down along x -axis, see the first arrow in Figure 10.

For adding a square on an L-shaped curve along an edge of length l , we first blow down the curve, where we take xy -coordinate such that y -axis is parallel and sufficiently close to the edge. After that, we perturb the curve near the multiple crossing at the origin, see the second arrow in Figure 10. By using some Δ -moves in Lemma 3.2(6), we can move the curve into the required L-shaped curve of the square added L-shaped region. The number of double points increases by $l(l-1)/2$. For the Kirby calculus, see the bottom figure in Figure 10 again. By the operation, the knot K changes by a right-handed full-twist along the unknot defined by the edge. If K is framed (i.e., with a surgery coefficient), the framing increases by the square of the linking number of K and u . The linking number is equal to the length l of the edge by Lemma 3.2(3).

4.2 Adding squares II

We can apply the operation adding squares twice successively by changing the edges, as in Figure 11. In the same figure, we also define a drawing notation. It is important which edge we apply the operation first.

Here, we state the effect of twice adding squares on the knots in S^3 . This is proved by Lemma 4.2.

Lemma 4.3 *Suppose that $P = X \cap \mathcal{L}$ be an L-shaped curve and the first edge e_1 and the second one e_2 of the region \mathcal{L} are specified. By $P'' = X \cap \mathcal{L}''$, we denote the resulting L-shaped curve obtained by adding n_1 squares along e_1 first and adding n_2 squares along e_2 second, successively. Then, the divide knot $L(P'')$ is equal to the knot K'' obtained by two twistings from $L(P)$ in S^3 in the following sense:*

First, we take the three component divide link $L(P \cup c_1 \cup c_2) = K \cup L_1 \cup L_2$ presented by the plane curve $P \cup c_1 \cup c_2$, where $L(P) = K$ and L_i is the component presented by slightly

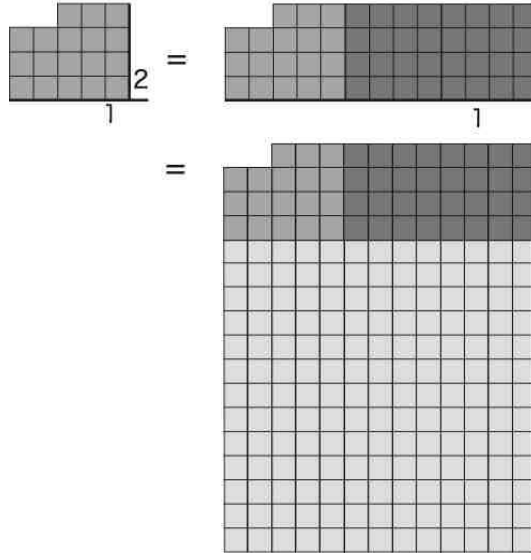


Figure 11: Adding squares II

pushed off c_i of e_i ($i = 1, 2$) into \mathcal{L} , see Figure 12. Note that $L_1 \cup L_2$ is a Hopf link. Next, we take n_1 full-twists of $K \cup L_2$ along L_1 . We call the resulting link $K' \cup L_2'$. Finally, we take n_2 full-twists of K' along L_2' . We call the resulting knot K'' .

In Lemma 4.3, n_1, n_2 are supposed to be positive, however, regarding the statement as *construction* of the knot K'' from $L(P)$ by two twistings, it works also in the case $n_1, n_2 < 0$. The knot K'' may be no longer a divide knot, by the obstruction of braid (quasi-)positivity of divide knots in Lemma 3.2(7) (and Lemma 3.5). In the next subsection, we consider the case K'' is the mirror image of an L-shaped divide knot.

4.3 Adding squares III

We extend the operation adding n squares into the case $n < 0$ partially, in analogy with Lemma 4.3. It corresponds to negative (i.e., left-handed) $|n|$ full-twists, and will be called “adding negative squares”. We consider the case that the resulting knot of negative full-twists of an L-shaped divide knot is the mirror image of another L-shaped divide knot under some conditions.

Definition 4.4 (Adding negative squares in a certain case) Let \mathcal{L} be an L-shaped region of type $[a_1, a_2; b_1, b_2]$ with a specified edge e . We assume that the edge is the (bottom) one of length a_2 ,

$$b_2 = b_1 + 1 \quad \text{and} \quad |n|a_2 > b_1 + 1.$$

Only under this condition, we define *adding n squares with $n < 0$ along e* as that the resulting region is of type

$$[a_2 - a_1 + 1, a_2; |n|a_2 - b_1 - 1, |n|a_2 - b_1].$$

This operation should be regarded geometrically as follows, see Figure 13: We assume that the initial L-shaped region is at the origin as in Definition 3.3 (once forgetting the assumption

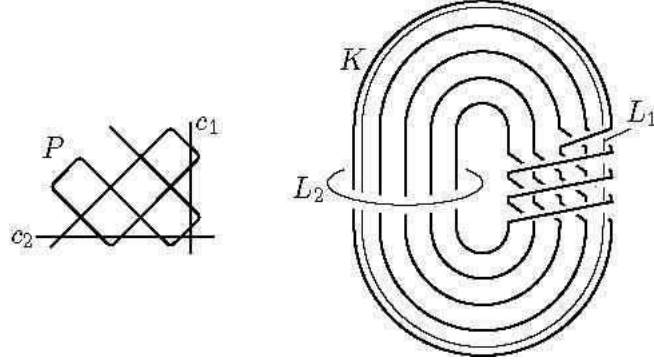


Figure 12: The effect of adding squares twice

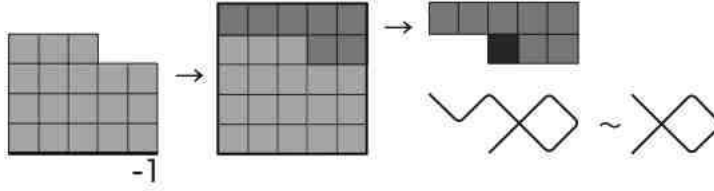


Figure 13: Adding negative squares

(*) to explain the operation by using xy -coordinate. Then adding n squares with $n < 0$ is defined as

$$\begin{aligned} & \text{cl}[\{(x, y) \mid 0 \leq x \leq a_2, 0 \leq y \leq |n|a_2\} \setminus \mathcal{L}] \\ & \cup \{(x, y) \mid a_1 - 1 \leq x \leq a_1, b_1 \leq y \leq b_1 + 1\}. \end{aligned}$$

By the finally added unit square at the concave point, if the concave point of the initial region is at an odd point, then that of the resulting region is also at an odd point, i.e., we can keep the condition (*).

Lemma 4.5 *Under the condition of adding negative squares in Definition 4.4, adding n squares with $n < 0$ on an L-shaped curve P along the edge corresponds to taking the mirror image of n right-handed (i.e., $|n|$ left-handed) full-twists on the divide knot $L(P)$ along the unknot defined by the edge.*

Proof. By the operation, the type of L-shaped curves is changed from $[a_1, a_2; b_1, b_1 + 1]$ to $[a_2 - a_1 + 1, a_2; |n|a_2 - b_1 - 1, |n|a_2 - b_1]$. By Lemma 3.5, the initial curve presents the closure of the braid $W(a_2)^{b_1}W(a_1)$ of index a_2 . The edge presents the braid axis in S^3 , The positive full-twist is $W(a_2)^{a_2}$ in this situation, and is in the center of the braid group. The resulting knot of the n full-twists is the closure of $W(a_2)^{b_1 + na_2}W(a_1) = W(a_2)^{-(|n|a_2 - b_1)}W(a_1)$. Note that $|n|a_2 > b_1 + 1$ is assumed. Lemma 4.5 follows from Lemma 3.8(−+). \square

Example 4.6 The example (from $[3, 5; 3, 4]$ to $[3, 5; 1, 2]$) in Figure 13 shows that the mirror image of $T(2, 3)$ is obtained by $P(-2, 3, 7)$ (in Example 3.6) by (-1) full-twist along the

unknot defined by the bottom edge, whose linking number with $P(-2, 3, 7)$ is ± 5 . Note that L-shaped curve of type $[3, 5; 1, 2]$ is the same curve with the billiard curve $B(2, 3)$ defined in Lemma 3.1.

Question 4.7 Extend the operation adding negative squares into (more) general cases.

4.4 How to construct the L-shaped curve

Preparation is complete. For each Berge's knot $K = K_{\mathcal{X}}(\delta, \varepsilon, A, k, t)$ in Type \mathcal{X} , we take the L-shaped region $\mathcal{L} = \mathcal{L}(\varepsilon, A, k, t)$ in Table 2, where we used the drawing notation of adding squares. Then, the plane curve $P = X \cap \mathcal{L}$, as a divide, presents the knot K or its mirror image. In fact, each L-shaped region in Table 2 is carefully constructed such that Berge's braid presentation of the knot in Lemma 2.1 agrees with the braid presentation of the region in Lemma 3.8 under the suitable choice of δ . The proof is in the next section.

In Table 2, we draw each L-shaped region in the case of the smallest A .

Demonstration. $K_{\text{III}}(\delta, \varepsilon, A, k, t)$ for $(\delta, \varepsilon, A, k, t) = (1, 1, 2, 2, 1)$.

By Table 1(1), $B = A(3 + 2k) - \varepsilon = 13$, $b = -\delta\varepsilon(2A + tB) = -17$. By Lemma 2.1, it has a braid presentation $W(13)^{-17}W(3)^1$. The surgery coefficient is $bB + \delta A = -219$. On the other hand, according to Table 2, the L-shaped region $\mathcal{L}(\varepsilon, A, k, t)$ with $(\varepsilon, A, k, t) = (1, 2, 2, 1)$ is the L-shaped region of type $[11, 13; 16, 17]$ (the region at the bottom in Figure 11), whose area is 219. By Lemma 3.5, its corresponding plane curve presents the closure of $W(13)^{16}W(11)$. By Lemma 2.3, the knot is equal to the closure of $W(13)^{17}W(3)^{-1}$, which is the mirror image of the required knot.

5 Proof of Theorem 1.3 and 1.4

Theorem 1.3 is proved by verifying that Berge's braid presentation of the knot (Lemma 2.1) and that of the L-shaped region in Table 2 (Lemma 3.8) agree, under the suitable choice of the sign δ in each Type. In the proof below, we will also decide the choice of δ (depending on \mathcal{X} and (ε, A, k, t)). We denote the result by $\delta_{\mathcal{X}}(\varepsilon, A, k, t)$, or $\delta_{\mathcal{X}}$ for short. It will be used in the proof of Lemma 5.1

Proof. (of Theorem 1.3) Here, we prove Theorem 1.3 only in the case of Type III. The proofs in the other cases are similar.

In Type III, in Table 1(1) and (2), we find

$$a = 0, \quad B = Al - \varepsilon, \quad l = 3 + 2k, \quad b = -\delta\varepsilon(2A + tB).$$

Thus, Berge's braid presentation $W(B)^bW(A+1-a)^\delta$ of the knot $K_{\text{III}}(\delta, \varepsilon, A, k, t)$ in Lemma 2.1 ([Bg2]) is

$$W((3 + 2k)A - \varepsilon)^{-\delta\varepsilon(2A + tB)} W(A + 1)^\delta \quad (1)$$

First, we consider the case $k = t = 0$.

Case 1+ (Type III, $\varepsilon = 1, k = t = 0$)

In Table 2, we find that the L-shaped region is of type $[2A - 1, 3A - 1; 2A - 1, 2A]$. On the other hand, Berge's presentation (1) is now $W(3A - 1)^{-2\delta A}W(A + 1)^\delta$. Here we choose $\delta = -1$. We use Lemma 3.8(+−) on the knot of $W(3A - 1)^{2A}W(A + 1)^{-1}$.

Case 1- (Type III, $\varepsilon = -1, k = t = 0$)

In Table 2, we find that the L-shaped region is of type $[A + 1, 3A + 1; 2A, 2A + 1]$. On the other hand, Berge's presentation (1) is now $W(3A + 1)^{2\delta A}W(A + 1)^\delta$. We choose $\delta = 1$. We use Lemma 3.8(++).

Next, we consider the case $k > 0$ ($t = 0$). In this case, we use the symmetry of Corollary 3.7 to verify that the parameter k contributes as the k full-twists on the knots.

Case 2+ (Type III, $\varepsilon = 1, k > 0, t = 0$)

The L-shaped region in Table 2 is of type $[2A - 1 + 2kA, 3A - 1 + 2kA; 2A - 1, 2A]$, which presents the closure of $W(3A - 1 + 2kA)^{2A-1}W(2A - 1 + 2kA)$ by Lemma 3.5. By the symmetry of Corollary 3.7, it presents the same knot of

$$W(2A)^{2A-1+2kA}W(2A - 1)^A = (W(2A)^{2A})^k W(2A)^{2A-1}W(2A - 1)^A.$$

Its first part $(W(2A)^{2A})^k$ means k full-twists. On the other hand, Berge's presentation (1) is now $W(3A - 1 + 2kA)^{-2\delta A}W(A + 1)^\delta$. We choose $\delta = -1$. We use Lemma 3.8(+−).

Case 2- (Type III, $\varepsilon = -1, k > 0, t = 0$)

The L-shaped region in Table 2 is of type $[A + 1, 3A + 1 + 2kA; 2A, 2A + 1]$, which presents the closure of $W(3A + 1 + 2kA)^{2A}W(A + 1)$ by Lemma 3.5. By Corollary 3.7, it presents the same knot of

$$W(2A + 1)^{A+1}W(2A)^{A+2kA} = W(2A + 1)^{A+1}W(2A)^A (W(2A)^{2A})^k.$$

Its final part $(W(2A)^{2A})^k$ means k full-twists of $2A$ strings of the braid of index $2A + 1$. On the other hand, Berge's presentation (1) is now $W(3A + 1 + 2kA)^{2\delta A}W(A + 1)^\delta$ by (1). We choose $\delta = 1$. We use Lemma 3.8(++).

Before we go into the case $t \neq 0$, we remark the followings.

- (1) The parameter t can be negative.
- (2) The parameters A, k and B are independent from t .
- (3) The parameter B is equal to the length of the edge that is added t squares.
- (4) In every case in Table 2, we can apply the operation adding t squares along the edge, i.e., the condition " $b_2 = b_1 + 1$ and $|t|a_2 > b_1 + 1$ " for adding negative squares are satisfied, even if $t = -1$.

From now on, we use Berge's braid presentation of $K_{\text{III}}(\delta, \varepsilon, A, k, t)$ in Lemma 2.1 ([Bg2]) in the form

$$W(B)^{-\delta\varepsilon(2A+tB)}W(A + 1)^\delta. \quad (2)$$

Case 3+ (Type III, $\varepsilon = 1, t > 0$)

The L-shaped region in Table 2 is of type $[B - A, B; 2A - 1 + tB, 2A + tB]$ with $B = 3A - 1 + 2kA$. On the other hand, Berge's presentation (2) is now $W(B)^{-\delta(2A+tB)}W(A + 1)^\delta$. We choose $\delta = -1$. We use Lemma 3.8(+−).

Case 3- (Type III, $\varepsilon = -1, t > 0$)

The L-shaped region in Table 2 is of type $[A + 1, B; 2A + tB, 2A + 1 + tB]$ with $B = 3A + 1 + 2kA$.

On the other hand, Berge's presentation (2) is now $W(B)^{\delta(2A+tB)}W(A+1)^\delta$. We choose $\delta = 1$. We use Lemma 3.8(++).

Before we consider the case $t < 0$, we remark the condition $2A \leq B$ in Section 2. In fact, $2A = B$ never occurs.

Case 4+ (Type III, $\varepsilon = 1, t < 0$) We use adding negative squares.

The type of L-shaped region in Table 2 is $[A + 1, B; |t|B - 2A, |t|B - 2A + 1]$ with $B = 3A - 1 + 2kA$. On the other hand, Berge's presentation (2) is now $W(B)^{-\delta(2A+tB)}W(A+1)^\delta$. We choose $\delta = 1$. We use Lemma 3.8(+++) on $W(B)^{|t|B-2A}W(A+1)$.

Case 4- (Type III, $\varepsilon = -1, t < 0$) We use adding negative squares.

The type of L-shaped region in Table 2 is $[B - A, B; |t|B - 2A - 1, |t|B - 2A]$ with $B = 3A + 1 + 2kA$, which presents the closure of $W(B)^{|t|B-2A-1}W(B-A)$. On the other hand, Berge's presentation (2) is now $W(B)^{\delta(2A+tB)}W(A+1)^\delta$. We choose $\delta = -1$. We use Lemma 3.8(+--) on the knot of $W(B)^{|t|B-2A}W(A+1)^{-1}$.

After all, for given parameters (ε, A, k, t) , we have shown that the L-shaped curve $P = X \cap \mathcal{L}$ of the L-shaped region $\mathcal{L} = \mathcal{L}_{\text{III}}(\varepsilon, A, k, t)$ in Table 2 presents $K_{\text{III}}(\delta, \varepsilon, A, k, t)$ for suitable choice of δ .

The proof of Theorem 1.3 in the case of Type III is completed. The cases of the other Types are proved by the same argument. \square

In the proof of Theorem 1.3, $\delta_{\text{III}}(\varepsilon, A, k, t)$, the suitable choice of δ in Type III is determined. Considering the other Types, it extends as

$$\delta_{\mathcal{X}}(\varepsilon, A, k, t) = \begin{cases} -1 & \text{if } \varepsilon \cdot \text{sgn}(t) = +1 \\ 1 & \text{if } \varepsilon \cdot \text{sgn}(t) = -1 \end{cases},$$

where $\text{sgn}(t) = 1$ if $t > 0$ or $t = 0$, $\text{sgn}(t) = -1$ otherwise. It means simply

$$-\delta_{\mathcal{X}} \cdot \varepsilon \cdot \text{sgn}(t) = +1. \quad (3)$$

Using $\delta_{\mathcal{X}}$, Theorem 1.3 means simply

$$L(P_{\mathcal{X}}(\varepsilon, A, k, t)) = K_{\mathcal{X}}(\delta_{\mathcal{X}}, \varepsilon, A, k, t).$$

Now, we consider the coefficients of the lens space surgeries. Before we start the proof of Theorem 1.4, we show

Lemma 5.1 *The surgery coefficient of $L(P_{\mathcal{X}}(\varepsilon, A, k, t))$ is positive.*

Proof. As in the proof of Theorem 1.3, here we prove the lemma only in the case of Type III. In Table 1(3), we find that the coefficient of $L(P_{\text{III}}(\varepsilon, A, k, t)) = K_{\text{III}}(\delta_{\mathcal{X}}, \varepsilon, A, k, t)$ (as in Type III) is

$$-\delta_{\mathcal{X}} \cdot \varepsilon \cdot (6A^2 - 3\varepsilon A + k(2A)^2 + tB^2).$$

Since $|B| = (3 + 2k)A - \varepsilon$ (the length of the longest edge), it holds that $\text{sgn}(6A^2 - 3\varepsilon A + k(2A)^2 + tB^2) = \text{sgn}(t)$. Thus the sign of the coefficient is equal to $-\delta_{\mathcal{X}} \cdot \varepsilon \cdot \text{sgn}(t) = +1$ by (3). The proof of the other Types are similar. \square

Question 5.2 Prove Lemma 5.1 without Berge's braid presentation. If a knot K is a closure of a positive braid and admits a lens space surgery, is the coefficient positive?

Now, we prove a precise version of Theorem 1.4 on the difference between the area $\text{area}(P)$ of the L-shaped curve and the surgery coefficient $\text{coef}(L(P))$ (> 0) of the lens space surgery as in Type \mathcal{X} .

Lemma 5.3 *Under the correspondence in Theorem 1.3, it holds that*

$$\text{area}(P) - \text{coef}(L(P)) = \begin{cases} 0 & \text{if } (-1)^a \cdot \varepsilon \cdot \text{sgn}(t) = +1 \\ 1 & \text{if } (-1)^a \cdot \varepsilon \cdot \text{sgn}(t) = -1 \end{cases}.$$

where $\text{sgn}(t) = 1$ if $t > 0$ or $t = 0$, $\text{sgn}(t) = -1$ otherwise. See Table 1(1) for the definition of a ($= 0$ or 1).

Proof. First, in the case $k = t = 0$, it is easy to verify the equation in Table 2. The parameters k and t with $t > 0$ contribute as adding positive squares. In the operation adding a positive square along an edge of length x , the area increases by x^2 . It is compatible with the terms $+kA^2 + tB^2$ (or $+k(2A)^2 + tB^2$) in the surgery coefficients in Table 1(3). In the case $t < 0$, we do the operation adding negative squares. Suppose that we get the curve P_{new} from P_{old} by adding t squares with $t < 0$ along an edge of length x . Then

$$\text{coef}(L(P_{\text{new}})) = -(\text{coef}(L(P_{\text{old}})) - |t|x^2),$$

since the new divide knot $L(P_{\text{new}})$ is the mirror image of the knot obtained by left-handed $|t|$ twists from $L(P_{\text{old}})$. On the other hand,

$$\text{area}(P_{\text{new}}) = |t|x^2 - \text{area}(P_{\text{old}}) + 1,$$

where the last $+1$ corresponds to the finally added unit square. Thus

$$\text{area}(P_{\text{new}}) - \text{coef}(L(P_{\text{new}})) = 1 - (\text{area}(P_{\text{old}}) - \text{coef}(L(P_{\text{old}}))).$$

We have the lemma. \square

Note that Berge's constant a in [Bg2, Table 3(p.15)] was defined geometrically in the context of doubly-primitive knots. It might be curious that a is related to the difference between the area and the coefficient as above.

Goda and Teragaito [GT] conjectured an inequality

$$2g(K) + 8 \leq |r| \leq 4g(K) - 1,$$

on the surgery coefficient r and the genus $g(K)$ of the hyperbolic lens space surgery (K, r) . It is called "Goda–Teragaito conjecture". We are concerned with the left hand-side inequality. By Proposition 3.4 and Lemma 5.3, we have:

Corollary 5.4 *Let $[a_1, a_2; b_1, b_2]$ be the type of the L-shaped curve of our presentation (in Table 2) of Berge's knot in Type III, IV, V and VI as a divide knot $L(P)$. Then, it holds that*

$$\text{coef}(L(P)) - 2g(L(P)) = a_2 + b_2 - 1, \text{ or } a_2 + b_2 - 2.$$

The parameters A, k or $|t|$ are greater, the difference $|r| - 2g(K)$ can be greater.

6 Further Observation

Divide presentation of L-shaped divide knots helps us to study the constructions of the knots, and the relationship among the lens space surgeries.

6.1 Twisted torus knots

Following Dean [D], by a *twisted torus knot* $T(p, q; r, s)$, if $r < p$, we mean the knot obtained from the torus knot $T(p, q)$ by s full-twists of r strings in the p parallel strings of $T(p, q)$ in the standard position. On the other hand, if $r > p$, we mean the knot obtained from the torus knot $T(p, q)$ as a closure of the braid $w(p, q)\sigma_p\sigma_{p+1}\cdots\sigma_{r-1}$ by s full-twists of all r strings, where $w(p, q)\sigma_p\sigma_{p+1}\cdots\sigma_{r-1}$ is a positive Markov stabilization of the standard braid $w(p, q)$ of $T(p, q)$ of index p to a braid of index r . The following lemma follows from the braid presentation in Lemma 3.5 (and Lemma 4.2).

Lemma 6.1 *Let (p, q) be a coprime pair of positive integers, and r, s integers satisfying $0 < r \neq p$ and $s > 0$. The twisted torus knot $T(p, q; r, s)$ is one of A'Campo's divide knots, and can be presented by an L-shaped curve of type*

$$\begin{cases} [q, q + rs; r, p] & \text{if } r < p \\ [rs + 1, q + rs; p, r] & \text{if } r > p \end{cases},$$

as A'Campo's divide knots, see Figure 14.

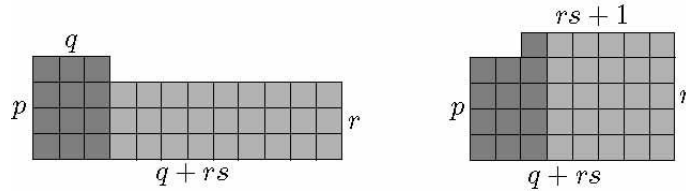


Figure 14: Twisted torus knots $T(p, q; r, s)$ (ex. $T(4, 3; 3, 3)$ and $T(4, 3; 5, 1)$)

Note that, a twisted torus knot $T(p, q; r, s)$ can be accidentally non-hyperbolic, a torus knot $T(p', q')$, or a cable knot $C(T(p', q'); m', n')$ of a torus knot, see [Y2, MY] for such phenomena.

Lemma 6.2 *Each knot in the following list is a twisted torus knot:*

(knots with $t = 0$)

$$\begin{aligned} K_{III}(1, -1, A, k, 0) &= T(2A + 1, A + 1; 2A, k + 1), \\ K_{IV}(-1, 1, A, k, 0) &= T(A, kA + (3A + 1)/2; A - 1, 1), \\ K_V(-1, 1, A, k, 0) &= T(A, (k + 1)A + 2; A - 1, 1), \\ K_{VI}(1, -1, A, 0, 0) &= T(A - 1, A + 1; A, 1), \end{aligned}$$

(knots with $t = -1$)

$$\begin{aligned} K_{III}(1, 1, A, k, -1) &= T(A, A + 1; A - 1, k + 2), \\ K_{IV}(-1, -1, A, k, -1) &= T((3A - 1)/2, A; (3A + 1)/2, k + 1), \\ K_V(-1, -1, A, k, -1) &= T(2A - 2, A; 2A - 1, k + 1). \end{aligned}$$

Proof. Only we have to do is to verify them by comparing types of the L-shaped regions, thus we omit the proof in detail. See Figure 15 for the case of $K_V(-1, -1, A, k, -1)$, which needs adding t square with $t = -1$. \square

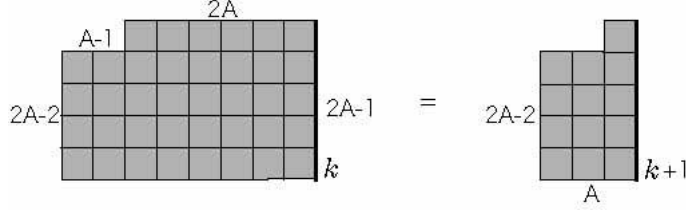


Figure 15: $K_V(-1, -1, A, k, -1) = T(2A - 2, A; 2A - 1, k + 1)$

Now, we count s full-twists along a fixed unknot as one twisting. If after s_1 full-twists along an unknot we take another s_2 full-twists along a different unknot, then we count the operation as two twistings.

Corollary 6.3 ([DMM]) *Every knot in Berge's in Type III, IV, V and VI is obtained at most one (s') full-twists from a twisted torus knot $T_w := T(p, q; r, s)$, thus is obtained at most two twistings from a torus knot, such that every knot in the twisting process ($T(p, q; r, i)$ with $0 \leq i \leq s$ and j full-twists of T_w with $0 \leq j \leq s'$) admits lens space surgery. Furthermore, as a twisted torus knot above, we can take $T(p, q; r, s)$ that satisfies $|r - p| = 1$.*

In the process of preparation of this paper, the author was informed by Motegi ([DMM]) that every knot in Berge's list (including Type VII, ..., XII, see Section 1) is obtained by at most two twistings from a torus knot, which includes the above Corollary.

6.2 Relations between different Types

See the L-shaped region in the left top figure in Figure 16. It is of type $[3, 5; 3, 4]$, and presents $K_{III}(-1, 1, 2, 0, 0)$ ($= P(-2, 3, 7)$). The L-shaped regions obtained by adding k squares along the right edge (denoted by R), we have a subsequence $K_{III}(-1, 1, 2, k, 0)$ in Type III, see Type III($\varepsilon = 1$) in Table 2. On the other hand, those obtained by adding $k + 1$ squares with $k \geq 0$ along the left edge (denoted by L), we have another subsequence $K_V(1, 1, 3, k, 0)$ in Type V (a different Type), see Type V($\varepsilon = -1$) in Table 2. By

$$K_{III}(-1, 1, 2, 0, 0) \mapsto K_V(1, 1, 3, k, 0),$$

we denote such a relation, regarding it as $K_{III}(-1, 1, 2, 0, 0) = "K_V(1, 1, 3, -1, 0)"$. Using L-shaped curve presentation in Table 2, we can see such relations more:

- For each A with $A \geq 3$, $K_{III}(-1, 1, A, 0, 0) \mapsto K_{IV}(1, -1, 2A - 1, k, 0)$.
- For each A , $K_{III}(1, -1, A, 0, 0) \mapsto K_{IV}(-1, 1, 2A + 1, k, 0)$.

Question 6.4 Find such relations more, especially in the case $t \neq 0$.

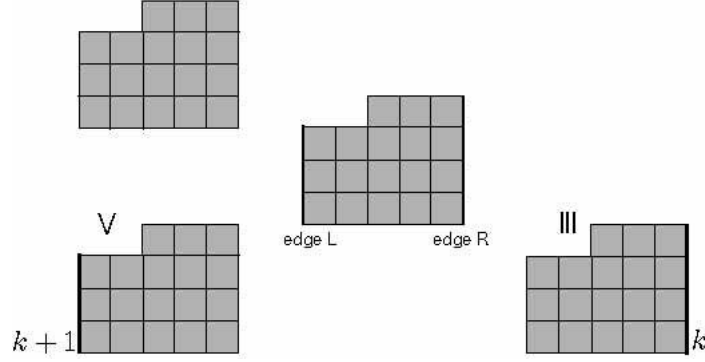


Figure 16: Type III and Type V

6.3 Parameters Translation

Parameters of Berge's knots are different among some papers. In Table 1(4), we give the translation formula between

$$(A, k) \text{ in [Bg2] and this paper} \quad \text{and} \quad (n, p) \text{ in [Bg, Ba, Ba3, DMM]}.$$

The signs δ, ε are commonly used among these papers.

For example, $K_{IV}(\delta, -1, 5, 1, t)$ (i.e., $(\varepsilon, A, k) = (-1, 5, 1)$) in this paper is, up to mirror image, obtained from the knot of $(\varepsilon, n, p) = (-1, 2, 4)$ in Type IV in [Bg, Ba, Ba3, DMM] by $\pm t$ full-twists.

The parameter p is defined as a positive integer, and our k is just a parallel shift of p such that " $k = 0$ at the minimal possible value as p ": the statement " $\varepsilon p \neq -2, -1, 0, 1$ " (see Table 1(4)) is referred as

$$"p \geq 2 \text{ if } \varepsilon = +1 \text{ and } p \geq 3 \text{ if } \varepsilon = -1" \quad \text{in [Bg, Ba, Ba3, DMM]}.$$

On the other hand, we say

$$"k := p - 2 \text{ if } \varepsilon = +1 \text{ and } k := p - 3 \text{ if } \varepsilon = -1", \quad \text{in the present paper.}$$

Acknowledgement. The author would like to thank to Professor Mikami Hirasawa, Dr. Tomomi Kawamura, Dr. Masaharu Ishikawa, Professor Sergei Chmutov, and Professor Norbert A'Campo for informing him on A'Campo's divide knot theory. The author also would like to thank to Professor Kimihiko Motegi, Professor Masakazu Teragaito, Professor Hiroshi Goda, Professor Noriko Maruyama, Dr. Toshio Saito, Dr. Kenneth Baker, Dr. Arnaud Deruelle, Dr. Hiroshi Matsuda, and Professor John Berge for helpful suggestion on lens space surgery.

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	a	A	l	$\delta, \varepsilon \in \{\pm 1\}, t \in \mathbf{Z}$	
III	0	$2, 3, 4, \dots$	$3, 5, 7, \dots$ (odd)	$B = Al - \varepsilon,$	$b = -\delta\varepsilon(2A + tB)$
IV	1	$5, 7, 9, \dots$ (odd)	$5, 7, 9, \dots$ (odd)	$B = (Al - \varepsilon)/2,$	$b = -\delta\varepsilon(A + tB)$
V	1	$3, 5, 7, \dots$ (odd)	$2, 3, 4, \dots$	$B = Al + \varepsilon^*,$	$b = -\delta\varepsilon(A + tB)$
VI	0	$4, 6, 8, \dots$ (even)		$B = 2A + 1,$	$b = \delta(A - 1 + tB)$

Here $\varepsilon^* \neq -1$ if $l = 2$ (since $0 < 2A \leq B$).

(1) The parities and ranges of A, l and the settings of a, B, b ([Bg2])

$$\begin{array}{lll}
\text{III: } l = 3 + 2k, & \text{V: } l = 2 + k \text{ (if } \varepsilon = +1), & \text{VI: } k \equiv 0 \text{ (} \varepsilon \equiv -1). \\
\text{IV: } l = 5 + 2k, & l = 3 + k \text{ (if } \varepsilon = -1), &
\end{array}$$

(2) Parameter k

$$\begin{array}{ll}
\text{III.} & -\delta\varepsilon (6A^2 - 3\varepsilon A + k(2A)^2 + tB^2), \\
\text{IV.} & -\delta\varepsilon \left(\frac{5}{2}A^2 - \frac{3}{2}\varepsilon A + kA^2 + tB^2 \right), \\
\text{V.} & -\delta (2A^2 + kA^2 + tB^2) \text{ if } \varepsilon = +1, \\
& \delta (3A^2 + kA^2 + tB^2) \text{ if } \varepsilon = -1, \\
\text{VI.} & \delta ((2A^2 - 1) + tB^2).
\end{array}$$

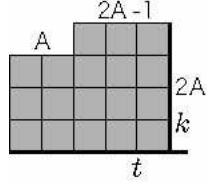
(3) Surgery coefficient: $bB + \delta A$

	A	k (if $\varepsilon = +1$)	k (if $\varepsilon = -1$)	$\varepsilon p \neq$
III	$n + 1$	$p - 1$	$p - 2$	$-1, 0$
IV	$2n + 1$	$p - 2$	$p - 3$	$-2, -1, 0, 1$
V	$2n + 3$	$p - 2$	$p - 3$	$-2, -1, 0, 1$
VI	$2n + 2$	-	(0)	-

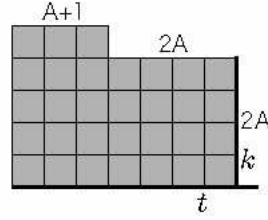
(4) Parameter Translation (to [Bg, Ba, Ba3, DMM]), see Subsection 6.3.

Table 1: Parameters of Berge's knots

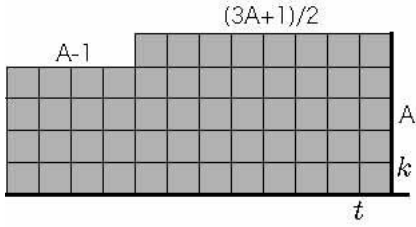
III. $\varepsilon = 1, A = 2, 3, 4, \dots$
 $\text{coef} = 6A^2 - 3A,$
 $\text{area}(\mathcal{L}) = 6A^2 - 3A,$



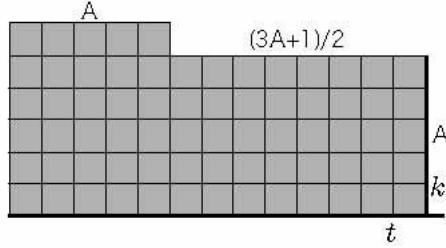
$\varepsilon = -1, A = 2, 3, 4, \dots$
 $\text{coef} = 6A^2 + 3A$
 $\text{area}(\mathcal{L}) = 6A^2 + 3A + 1$



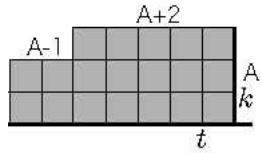
IV. $\varepsilon = 1, A = 5, 7, 9, \dots$
 $\text{coef} = \frac{5}{2}A^2 - \frac{3}{2}A,$
 $\text{area}(\mathcal{L}) = \frac{5}{2}A^2 - \frac{3}{2}A + 1,$



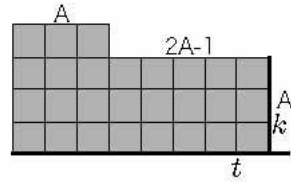
$\varepsilon = -1, A = 5, 7, 9, \dots$
 $\text{coef} = \frac{5}{2}A^2 + \frac{3}{2}A$
 $\text{area}(\mathcal{L}) = \frac{5}{2}A^2 + \frac{3}{2}A$



V. $\varepsilon = 1, A = 3, 5, 7, \dots$
 $\text{coef} = 2A^2,$
 $\text{area}(\mathcal{L}) = 2A^2 + 1,$



$\varepsilon = -1, A = 3, 5, 7, \dots$
 $\text{coef} = 3A^2$
 $\text{area}(\mathcal{L}) = 3A^2$



VI. $A = 4, 6, 8, \dots$
 $\text{coef} = 2A^2 - 1$
 $\text{area}(\mathcal{L}) = 2A^2$

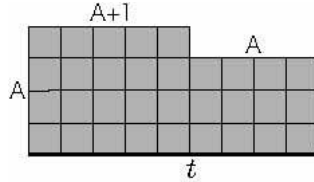


Table 2: Berge's knots presented by L-shaped regions